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The world viewed from outside

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Abstract

Simple models of the cosmos are given by space-forms of a stretched 3-sphere. The isometric embedding of these manifolds into higher-dimensional Euclidean spaces is an unsolved problem that has yielded some partial solutions of bizarre beauty.

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1. Introduction. The S^3

The idea that space is finite is at least as old as the ancient Greek philosophers. However, if they also believed in R^3 of Euclid's geometry, they were suffering from schizophrenia. The paradigm of a compact world is the S^3, the sphere in four dimensions. It is easily imagined as an analogon to an intrinsic description of the S^2.

One can visualize the S^2 by playing Santa describing co-latitude circles about his home. Their circumference increases less than linearly with the radius, reaches a maximum at the equator and shrinks thereafter finally to zero at the south pole.

According to the Swiss mathematician Andreas Speiser [31] it was Dante Alighieri who envisioned the S^3 as a model of his hell-centered medieval universe. Santa is replaced by the prince of darkness in the center of the earth . The co-latitude circles are now upgraded in dimension to concentric spheres like the surface of terra followed by those of the seven planets and the outermost of the fixed stars. But beyond this sphere of maximal area Dante

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saw further orbs in el paradiso populated by good souls, saints and angels and decreasing in area with increasing distance and holiness – ending in the most distant sphere shrunk to a point: God the antipode of the devil. Tintoretto painted this heavenly scene a la Dante. When you visit Paris, look it up in the Louvre!

In exact mathematical form, free of theology, the S^3 appears in the middle of the 19th century. The Swiss high school teacher Ludwig Schläfli embeds it into a four-dimensional Euclidean space R^4 with metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}u^2 \tag{1.1}$$

as a hypersurface through the equation

$$x^2 + y^2 + z^2 + u^2 = R^2 = \mathrm{const} > 0. \tag{1.2}$$

But his theory of *manifold continuity* is published only half a century later [28].

Introducing hyperspherical coordinates into R^4 we have

$$x + \mathrm{i}y = R \sin\chi \sin\theta\, \mathrm{e}^{\mathrm{i}\phi}, \quad z = R\sin\chi\cos\theta, \quad u = R\cos\chi. \tag{1.3}$$

We calculate the differentials $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, and $\mathrm{d}u$, substitute into Schläfli's four-dimensional Pythagoras (1.1) and obtain for $\mathrm{d}R = 0$

$$\mathrm{d}s^2 = R^2(\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)), \tag{1.4}$$

the intrinsic line element of the S^3 with radius R in polar coordinates.

In Dante's universe the devil sits at $\chi = 0$. A 2-sphere centered about him with the radius $R\chi$ has area $4\pi(R\sin\chi)^2$. The 9*th* heaven, the primum mobile, has radius $\frac{1}{2}R\pi$. At $\chi > \frac{1}{2}\pi$ we are in the empyreum, the abode of god, and the Almighty thrones at $\chi = \pi$.

We know precisely the date when the *intrinsic* description of the S^3 was promulgated. It was the Saturday before Pentecost 1854 around noon. The 77-year old Gauss, near death, had summoned his 28-year old postdoc Bernhard Riemann and told him to give a lecture to the philosophical faculty of Göttingen University the following day – *to get the thing over with*, as Gauss said. What he heard that day must have stunned him. In his unpublished papers Gauss had foreseen much of the mathematics in his century and he was reluctant to praise anything he had discovered himself. Riemann's student Richard Dedekind tells us that after the lecture Gauss walked home with his physicist friend Wilhelm Weber praising the depth of Riemann's ideas with an excitement unheard of from the prince of the mathematicians. [25]

Riemann's lecture in which he had developed the intrisic Riemannian geometry was published in 1868, two years after his death. William Kingdon Clifford translated it immediately into English and made a singular discovery about Riemann's S^3 [5].

So far we have looked at the S^3 as two solid three-dimensional balls: the *northern* ball defined by the range of the radial coordinate χ from zero to $\frac{1}{2}\pi$ and the *southern* ball defined by the range from $\frac{1}{2}\pi$ to π. The two solid balls are glued together at the 2-sphere $\chi = \frac{1}{2}\pi$, the surfaces of these two balls. This is completely analogous to imagining the 2-sphere S^2 consisting of two discs, the northern and the southern hemispheres glued together at the equator. Clifford looked at the S^3 in a different way that has no analog in the S^2. His discovery is best described by combining the real coordinates x and y into the complex number z_1 and z with u into the complex number z_2

$$x + \mathrm{i}y \equiv z_1, \quad z + \mathrm{i}u \equiv z_2. \tag{1.5}$$

We can then write the equation of the S^3 as

$$x^2 + y^2 + z^2 + u^2 = z_1\bar{z}_1 + z_2\bar{z}_2 = R^2, \tag{1.6}$$

where a bar indicates the complex conjugate. The metric in R^4 becomes then correspondingly

$$\mathrm{d}s^2 = \mathrm{d}z_1\,\mathrm{d}\bar{z}_1 + \mathrm{d}z_2\,\mathrm{d}\bar{z}_2. \tag{1.7}$$

Complex numbers can also be represented through polar coordinates in their Argand planes:

$$z_1 = r_1 \mathrm{e}^{\mathrm{i}\phi_1}, \quad z_2 = r_2 \mathrm{e}^{\mathrm{i}\phi_2}, \quad \phi_j \bmod 2\pi, \quad j = 1, 2. \tag{1.8}$$

The S^3 is thus given by the equation

$$r_1^2 + r_2^2 = R^2. \tag{1.9}$$

Taking r_1 constant fixes r_2 – it has to be positive and defines a two-dimensional surface, parametrized by the real coordinates ϕ_1 and ϕ_2. This gives a map of a square $2\pi \times 2\pi$ into the S^3 (see Fig. 1).

Opposite edges of the square have to be identified because of the periodicity of the exponential functions. By identifying the upper and the lower edge we turn the square into a cylinder and the left and the right edges into circles. When we glue these circles together we make the cylinder topologically into a torus. The surface

$$r_1 = a = \text{const}, \quad 0 < a < R, \tag{1.10}$$

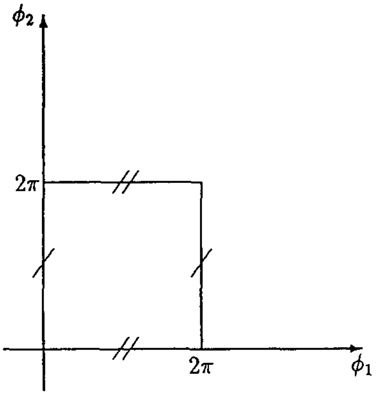

Fig. 1.

is thus a two-dimensional torus. The points with $r_1 < a$ form the interior of a solid bagel and, surprisingly, the points with $r_1 > a$ form the interior of another solid bagel since

$$r_2 = +\sqrt{R^2 - r_1^2} < \sqrt{R^2 - a^2} = \text{const.} \tag{1.11}$$

Clifford's S^3 consists thus of two solid bagels glued together over their common surface, called the Clifford surface. The real shock came when Clifford calculated the metric on his torus. With r_1 and r_2 being constant

$$\mathrm{d}z_1 = \mathrm{i}r_1 \mathrm{e}^{\mathrm{i}\phi_1}\,\mathrm{d}\phi_1, \qquad \mathrm{d}z_2 = \mathrm{i}r_2 \mathrm{e}^{\mathrm{i}\phi_2}\,\mathrm{d}\phi_2 \tag{1.12}$$

and, therefore, from (1.7)

$$\mathrm{d}s^2 = |\,\mathrm{d}z_1|^2 + |\,\mathrm{d}z_2|^2 = r_1^2\,\mathrm{d}\phi_1^2 + r_2^2\,\mathrm{d}\phi_2^2 \tag{1.13}$$

and with new coordinates

$$r_1\phi_1 \equiv \xi, \quad r_2\phi_2 \equiv \eta, \tag{1.14}$$

he obtained

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 + \mathrm{d}\eta^2. \tag{1.15}$$

This means: the Clifford surface is flat like the Euclidean plane and locally indistinguishable from it.

Clifford's discovery signalled a turning point in our thinking about the structure of space. Mathematicians realized that the local structure of space did not determine its properties in the large and was compatible with different topologies of the whole manifold.

Felix Klein discovered that a flat torus need not be orientable: the Klein bottle [13] – and found that the projective plane has the topology of a 2-sphere on which opposite points are identified. Wilhelm Killing then teacher at a Seminary for priests posed the problem of finding the possible forms of spaces with constant curvature (planes, spheres, and their hyperbolic analogs) [12]. He called it the Clifford–Klein space problem. He suggested that it could be solved by finding all the properly discrete subgroups of the group of motions of the space which – apart from the identity – leave no point fixed. The classification was up to affine transformations of the space forms.

The two-dimensional flat torus was obtained by identifying all points under the transformations

$$x' = x + 2\pi m, \quad y' = y + 2\pi n, \qquad m, n \in Z. \tag{1.16}$$

These transformations form a fixpoint-free subgroup of motions of the plane and the group is properly discrete since the orbit of a given point (x_0, y_0) containing all its transforms has no point of accumulation.

These were some of the early results [30]:

(a) The S^{2n} has as only other space form the RP^{2n}, the $2n$-dimensional real projective space, that is obtained from the S^{2n} by identifying opposite points.

(b) The Euclidean plane has four space forms beside the full plane itself, i.e., cylinder, Möbius band, torus, and Klein bottle.

(c) There exist also compact space forms of the flat three-dimensional space obtained by identifying all points under

$$\boldsymbol{x}' = \boldsymbol{x} + 2\pi \boldsymbol{m}, \qquad \boldsymbol{m} = (m_1, m_2, m_3), \quad m_j \in Z. \tag{1.17}$$

(d) The three-dimensional space of constant positive curvature has an infinite number of space forms. Typical examples are Killing's *lens spaces*. They are characterized by two relative prime positive integers p and q with $p > 1$. The points of the S^3 are given by the equation

$$|z_1|^2 + |z_2|^2 = R^2. \tag{1.18}$$

The transformations

$$z_1' = \exp\left(\frac{2\pi i n}{p}\right) z_1, \quad z_2' = \exp\left(\frac{2\pi i q n}{p}\right) z_2, \qquad n = 0, 1, \ldots, p-1, \tag{1.19}$$

are a properly discrete subgroup of the isometries for the space S^3 free of fixed points. We obtain the lens space $L(p, q)$ by identifying the p-tuples of points that are obtained through the action of the group on every point of the space. This quotient space has a volume that is p-times smaller than the volume of S^3. The simplest non-trivial example of a lens space is obtained for $p = 2,\ q = 1$. The elements of the subgroup are the two transformations

$$n = 0: \quad (z_1' = z_1, z_2' = z_2), \qquad n = 1: \quad (z_1' = -z_1, z_2' = -z_2). \tag{1.20}$$

The *lens* consists thus of the ball forming the Northern hyper-hemisphere of S^3 with opposite points on its surface identified. This space form is known as RP^3, the three-dimensional real projective space.

The lens spaces do not exhaust the possibilities for positive curvature. The great topologist Heinz Hopf gave exact proofs for Killing's theory of the space forms in his thesis in 1925 [10]. William Threlfall and Herbert Seifert listed in 1931–32 [34] all possibilities completely. Without exception the spaces are compact.

2. Models of the universe

Bernhard Riemann and William Kingdon Clifford had tried to link the curvature of space with the presence of matter and its gravitational field. Not much is known about these speculations. Both authors died at an early age of tuberculosis. After Riemann's lecture had been published physicists like Hermann von Helmholtz and astronomers like Karl Schwarzschild thought seriously that the physical space could have a curvature different from zero. From the measurement of star parallaxes Schwarzschild concluded around 1900 that if space was an S^3 its radius R had to be larger than 100 light years [29].

In 1916 Albert Einstein and David Hilbert independently found the equations that relate the curvature of space–time with the energy–momentum–stress tensor of matter in Einstein's theory of gravitation. In the following year Einstein proposed a model of the universe with the line element

$$ds^2 = -\,dt^2 + R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{2.1}$$

known as the Einstein cosmos. Its space sections for t = const were given by Riemann's S^3 with radius R. By suggesting that the space be compact he avoided the use of boundary conditions for his field equations. Since the matter was on the average at rest in S^3 he thought to have shown that the rotation of a body like the earth against Newton's absolute space was now a rotation against a physical agent, the matter in S^3. The physicist Ernst Mach had criticized Newton's dynamics by saying that rotation should only be conceived as rotation relative to the distant masses in the universe [17]. Einstein thought that his theory thus incorporated *Mach's principle*. When he visited Göttingen in 1918 the mathematician Felix Klein suggested to him that the space might not be an S^3 but an RP^3. Einstein reportedly caught on immediately by saying *you mean, it's only half as large*.

The spherical space S^3 and the *elliptical* space RP^3 are isotropic and homogeneous. No point is distinguished in these spaces as a center and no direction is singled out. The only other three-dimensional spaces with this property are the Euclidean R^3 and the hyperbolic space of constant negative curvature, both not compact.

The Russian meteorologist and mathematician Alexander Friedmann discovered in 1922 solutions for the Einstein field equations with matter in which the radius $R(t)$ was a function of the time t. They could describe an expanding universe. Based on the observations of the astronomers Vesto Slipher and Edwin Hubble the Belgian priest Georges Lemaître proposed in 1931 a singular beginning for Friedmann's model. It is widely known as the big bang theory of the universe and now probably taught in kindergarten.

Similar models of an expanding universe for zero and negative curvature of space were found after Friedmann's discovery. To this day astronomers have not been able to decide empirically which sign for the curvature of space was selected by God. Many cosmologists, trying to read God's mind – as one says nowadays – vote for positive curvature since space becomes then compact in a natural way. This might enable astronomers to study a *fair sample* of the cosmos – a hopeless task in a spatially infinite universe.

3. Bianchi's models

It is clear that the distribution of matter, therefore the space itself, in the real universe is not exactly isotropic and homogeneous. Thus the need for more general models of the universe. The great difficulties in finding solutions to the system of 10 highly non-linear partial differential equations of the second order in Einstein's theory led us to study more general models with still a high degree of symmetry, that reduce Einstein's equations to a system of ordinary differential equations. This can be achieved by keeping space homogeneous but relaxing the requirement of isotropy. The idea is to stretch the S^3 in different directions by different amounts but everywhere by the same amounts independent of position.

The concept of an anisotropic but homogeneous space has no analogon in two dimensions. For a 2-space to be homogeneous its Gaussian curvature has to be constant and that leads automatically to rotational invariance about every point, i.e. local isotropy. It was a major geometrical discovery in the last century when the Italian geometrician Luigi Bianchi classified the real three-dimensional Lie algebras [4] and found nine different types of 3-spaces in which their Lie groups act as isometries. Only Bianchi type IX leads naturally to a compact space and a deformation of the S^3.

To appreciate the exquisite beauty of Bianchi's construction we return to its paradigm, the S^3. Compared to S^2, hailed since Pythagoras as a symbol of perfection, the S^3 possesses an even higher symmetry as the only n-dimensional sphere that is itself a simple and semi-simple Lie group (the other simple but not semi-simple one is S^1, the circle). The group is known as SU_2, the group of 2×2 unitary matrices with determinant 1 which is isomorphic to the compact symplectic group $Sp(1)$, the group of unit quaternions. This group is also the covering group of $SO(3)$, the rotation group of the Euclidean R^3.

4. Hamilton's quaternions

Quaternions form a four-dimensional vector space over the real numbers. A quaternion ξ can be written in Hamilton's units as

$$\xi = u\,\mathrm{1} + x\,\mathrm{i} + y\,\mathrm{j} + z\,\mathrm{k}, \quad u, x, y, z \in R, \quad \mathrm{1}, \mathrm{i}, \mathrm{j}, \mathrm{k} \in H, \tag{4.1}$$

with

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{i\,j} = -\mathrm{j\,i} = \mathrm{k}, \quad \mathrm{j\,k} = -\mathrm{k\,j} = \mathrm{i}, \quad \mathrm{k\,i} = -\mathrm{i\,k} = \mathrm{j}. \tag{4.2}$$

They form a skew field and the conjugate quaternion $\bar{\xi}$ is defined by

$$\bar{\xi} = u\,\mathrm{1} - x\,\mathrm{i} - y\,\mathrm{j} - z\,\mathrm{k}. \tag{4.3}$$

The norm of the quaternion ξ is given by

$$\xi\bar{\xi} = \bar{\xi}\xi = u^2 + x^2 + y^2 + z^2, \tag{4.4}$$

and for two quaternions ξ and η one obtains

$$\overline{\xi\eta} = \bar{\eta}\bar{\xi}. \tag{4.5}$$

The points of S^3 can now be thought of as quaternions ξ with

$$\xi\bar{\xi} = R^2. \tag{4.6}$$

If we define two unit quaternions λ and ρ with

$$\lambda\bar{\lambda} = \rho\bar{\rho} = 1, \tag{4.7}$$

we can write the most general isometry of S^3 continuous with the identity as

$$\xi' = \lambda\xi\bar{\rho}. \tag{4.8}$$

It follows at once that

$$\xi'\bar{\xi}' = \xi\bar{\xi} = R^2. \tag{4.9}$$

Multiplication of ξ with the quaternion λ from the left gives rise to the group $Sp(1)$ of left translations while multiplication with $\bar{\rho}$ from the right leads to the right translations. The associative law of multiplication shows that the two groups commute with each other. While left and right translations separately make the S^3 homogeneous, isotropy about a point like $\xi_0 = R$ comes about through the diagonal of the direct product of the left and right-actions, namely by putting $\rho = \lambda$:

$$\xi' = \lambda\xi\bar{\lambda} \longrightarrow R = R. \tag{4.10}$$

While the poor S^2 cannot carry even one vector field or, its dual, a single differential 1-form, the S^3 has two ∞^3 sets of each. To construct this cornucopia of invariant structures one picks a vector or a form in one point of S^3 and moves it by left or right translations into the other points of the manifold. This turns out to be extremely simple and one creates in this way left- or right-invariant vector fields or forms on the manifold. If one puts the flow lines on, say, a left-invariant vector field one gets on the S^3 a system of great circles known as the Clifford left parallels. Projection along these great circles, known as the Hopf map, turns the S^3 into a principal fiber bundle over the base manifold S^2 [11].

To deal with the metric it is preferable to use left-invariant differential forms. On the S^3 given by

$$\xi\bar{\xi} = R^2, \quad \xi \in H \tag{4.11}$$

we take the differential forms

$$\omega = \frac{1}{2R}(\bar{\xi}\,\mathrm{d}\xi - \mathrm{d}\bar{\xi}\,\xi) = -\bar{\omega}. \tag{4.12}$$

Under left translation with fixed λ

$$\xi' = \lambda\xi, \quad \lambda\bar{\lambda} = 1, \qquad \lambda \in H, \tag{4.13}$$

we have

$$\omega' = \omega. \tag{4.14}$$

Because of $\bar{\xi}\,\xi = R^2$ we notice that

$$\mathrm{d}\bar{\xi}\,\xi + \bar{\xi}\,\mathrm{d}\xi = 0, \tag{4.15}$$

which gives at the point ξ_0

$$\omega|_{\xi=\xi_0} = \mathrm{d}x\,\mathrm{i} + \mathrm{d}y\,\mathrm{j} + \mathrm{d}z\,\mathrm{k}, \quad \xi_0 = R + 0\mathrm{i} + 0\mathrm{j} + 0\mathrm{k}. \tag{4.16}$$

We get for the metric at this point

$$\mathrm{d}s^2 = \mathrm{d}\xi\,\mathrm{d}\bar{\xi} = -\omega^2. \tag{4.17}$$

If we put

$$\omega = \omega_1 \mathrm{i} + \omega_2 \mathrm{j} + \omega_3 \mathrm{k}, \tag{4.18}$$

we can write for the left-invariant metric

$$\mathrm{d}s^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \equiv \omega_l \omega_l, \quad l = 1, 2, 3, \tag{4.19}$$

which becomes at ξ_0

$$\mathrm{d}s^2|_{\xi=\xi_0} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{4.20}$$

A simple calculation shows that the metric (4.19) is also invariant under the right translation

$$\xi' = \xi\bar{\rho}, \quad \rho\bar{\rho} = 1, \quad \mathrm{d}\xi'\,\mathrm{d}\bar{\xi}' = \mathrm{d}\xi\bar{\rho}(\overline{\mathrm{d}\xi\,\bar{\rho}}) = \mathrm{d}\xi\,\mathrm{d}\bar{\xi} = \mathrm{d}s^2 \tag{4.21}$$

since $\bar{\bar{\rho}} = \rho$ and $\bar{\rho}\rho = 1$. The left-invariant vector fields which take at ξ_0 the values

$$\omega_j(X_k) = \delta_{jk}, \quad X_1|_{\xi=\xi_0} = \frac{\partial}{\partial x}, \quad X_2|_{\xi=\xi_0} = \frac{\partial}{\partial y}, \quad X_3|_{\xi=\xi_0} = \frac{\partial}{\partial z} \tag{4.22}$$

are dual and orthogonal to the differential 1-forms ω_j $(j, k = 1, 2, 3)$. Their great circle flow lines turn the S^3 into a continuous Jungle Gym formed by three sets of Clifford left parallels orthogonal to the surface elements defined by the left-invariant 1-forms. Clifford parallels keep a constant distance from each other. They are geodesics in the S^3 and look locally like skew straight lines that try to diverge from each other but are bent together by the curvature of the space exactly compensating for their divergence. Any two parallels become linked like two rings always keeping a constant distance. The Clifford parallels with a constant distance a from a given geodesic fill the surface of a torus with this geodesic as circular axis. The left parallels wind around the torus meridionally once in the opposite way as the right parallels. Two left parallels and two right parallels on the torus form a parallelogram on this Clifford surface that develops into an ordinary parallelogram in the Euclidean plane.

5. The stretched S^3

With this sketch of the S^3 anatomy the structure of Bianchi's stretched S^3 is easily formulated. While keeping the whole scaffold for S^3 in place we vary the definition of the metric to

$$\mathrm{d}\tilde{s}^2 = \frac{1}{abc}(a\omega_1^2 + b\omega_2^2 + c\omega_3^2), \quad a, b, c > 0, \text{ const.} \tag{5.1}$$

Since the new metric on the S^3 is built upon the left-invariant differential forms it is still invariant under left translations and keeps the homogeneity intact. The space is stretched by factors proportional to $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in three orthogonal directions. All its curvature properties will not depend on position. Only if $a = b = c$ will isotropy be preserved. This is quite different from the stretching of the periodic cube in the Euclidean space into a brick; there the space remains flat and the local isotropy will not be affected.

The metric on the strained S^3 is a symmetric strain tensor on the 3-space of the left-invariant differential forms. The tensor stretches the unit sphere in the cotangent space of each point into an ellipsoid with the semi-axes proportional to $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ and we thus shall lose rotational invariance of the metric. If the ellipsoid is one of revolution, say, with $a = b$, the right translations

$$\rho = \cos\tfrac{1}{2}\phi + \sin\tfrac{1}{2}\phi k \equiv \exp(\tfrac{1}{2}\phi k), \quad \phi \in R, \ k \in H, \tag{5.2}$$

will give rotations about the c-axis which transform the ellipsoid – and thus the metric – into itself. We speak then of the symmetric case. If all axes are different, the only right translations surviving as isometries are (besides the identity) the three rotations by π of the ellipsoid about its three principal axes. These operations form the Klein 4-group.

It is the nature of S^3 as the compact, simple non-Abelian Lie group $Sp(1)$ which allows us to study the arcane notion of space curvature as mere *ether-curling*.

The S^3 of radius R has the metric

$$ds^2 = \omega_l \omega_l, \quad l = 1, 2, 3, \tag{5.3}$$

expressed in terms of the left-invariant differential forms ω_l. The size of space is fixed through the equations of Ludwig Maurer and Elie Cartan which express the differentials of the left-invariant forms ω_l again in terms of their cross vector products [15]

$$d\omega_l = -\frac{1}{R}\varepsilon_{lmn}\,\omega_m \wedge \omega_n, \quad l, m, n = 1, 2, 3, \ \ \varepsilon_{123} = +1. \tag{5.4}$$

Levi-Civita's ε_{lmn} form the completely skew-symmetric set of structure constants for the Lie algebra $sp(1)$. The Gaussian curvature K in any geodesic plane is given by

$$K = 1/R^2. \tag{5.5}$$

One can imagine that the three left-invariant vector fields X_k describe the three different ether streams blowing in different directions and interpenetrating each other. The differential forms ω_l measure the speeds of these ethereal winds and the angles they make with each other.

Assume now that the metric has the form (5.3). That means that the speeds ds/dt, or in modern notation,

$$\omega_l(d/dt) \tag{5.6}$$

are all equal to 1 and orthogonal. The Maurer–Cartan equations describe then how a feather wafting in one of the invisible streams will turn with respect to the two other streams. Its angular speed will be given by $1/R$. Local isotropy of space makes the little propeller turn in every direction at the same rate. It is the beautiful connection between the small and the large in a group like a simple S^3 that a wafting feather feels the size of the universe.

To study the curvature of a Riemannian manifold one needs further differentiation to construct the curvature tensor of Riemann and Christoffel. But for the space of a simple Lie group one does not need to go to such subtleties. Here the group introduces a metric. The metric and the curvature tensor are quadratic forms built from the structure constants of the group.

If we deform the metric of the S^3 with $R = 2$ by constants a, b, c as in (5.1), we shall keep fixed the differential forms ω_l and their structure constants in the Maurer–Cartan equations (5.4). Space will now be stretched by factors proportional to $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in orthogonal directions. This changes the speeds of the ether flow in our picture in general in a non-isotropic way. Correspondingly, the angular velocities of our little propellers will now become dependent on direction. The three great circle systems of Clifford left parallels will remain geodesic. Their length changed from 4π now to $4\pi/\sqrt{bc}$, $4\pi/\sqrt{ca}$, and $4\pi/\sqrt{ab}$. Also the volume of the stretched S^3 goes from $16\pi^2$ to $16\pi^2/abc$.

To make sure that the stretching of S^3 actually leads to an intrinsically changed manifold one calculates the Ricci tensor of the manifold. The eigenvalues of this tensor are invariants while the tensor describes completely the curvature properties of the three-dimensional Riemannian manifolds.

We obtain for the eigenvalues of the Ricci tensor

$$R_{11} = 2(s-b)(s-c), \quad R_{22} = 2(s-c)(s-a), \quad R_{33} = 2(s-a)(s-b) \tag{5.7}$$

with

$$s = \tfrac{1}{2}(a+b+c). \tag{5.8}$$

The eigenvectors of the tensor are in the direction of the principal strains $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. In the following we shall always choose

$$0 < a \leq b \leq c. \tag{5.9}$$

It will lead to

$$R_{11} \leq R_{22} \leq R_{33}, \quad R_{33} > 0. \tag{5.10}$$

It is then not difficult to see that different ratios $a : b : c$ lead to different ratios $R_{11} : R_{22} : R_{33}$ and vice versa in the allowed range of the eigenvalue ratios (except for $a + b = c$). (See Figs. 2 and 3.)

6. The dantes

CANTO XXVIII

THE NINTH SPHERE: THE PREMIUM MOBILE

The Angel Hierarchy

DANTE TURNS from Beatrice and beholds the vision of GOD AS A NON-DIMENSIONAL POINT OF LIGHT ringed by NINE GLOWING SPHERES representing the ANGEL HIERARCHY.

Dante is puzzled because the vision seems to reverse the order of the Universe, the highest rank of the angels being at the center and represented by the smallest sphere.

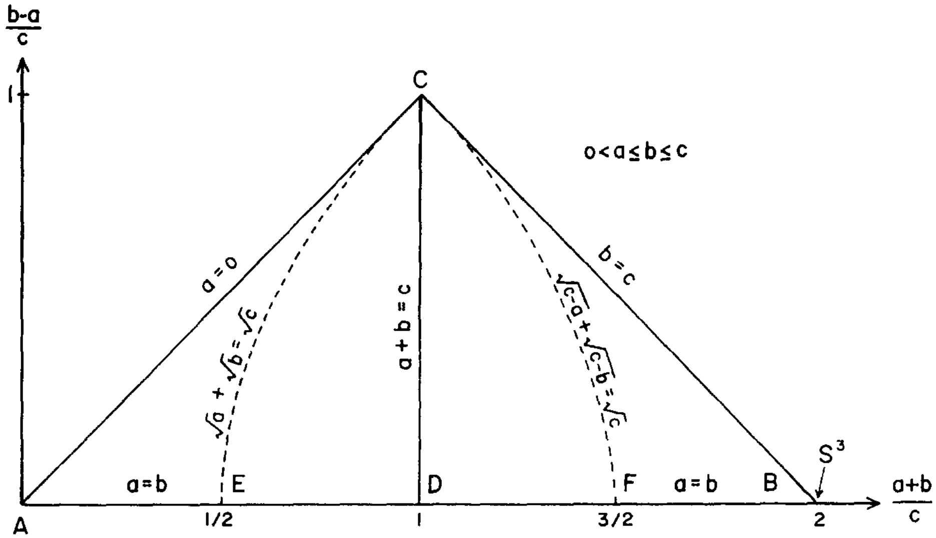

Fig. 2. Diagram for the stretching ratios of an S^3. The S^3-metric $ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ with $d\omega_j = -\frac{1}{2}\varepsilon_{jkl}\,\omega_k \wedge \omega_l$ is distorted into $d\tilde{s}^2 = (abc)^{-1}[a\omega_1^2 + b\omega_2^2 + c\omega_3^2]$. The hypotenuse of the right-angled triangle ABC covers symmetric snake-like shapes while the side BC describes turtle-like shapes. The side AC (with end points included) deals with degenerate configurations which are excluded. The vertex B denotes the isotropically stretched S^3. Here $a = b = c$. For the right triangle BCD all eigenvalues of the Ricci tensor are positive. In its right part BCF all principal curvatures are larger than zero. On the dashed line CF the smallest principal curvature vanishes. (This means that the largest eigenvalue of the Ricci tensor becomes the sum of the two other eigenvalues.) On the line CD the two smallest eigenvalues of the Ricci tensor take on the value zero. The principal curvatures are there: ab, ab, $-ab$. In the left triangle ACD the two smallest eigenvalues of the Ricci tensor are always negative while the largest eigenvalue remains positive. In the region CDE the Ricci scalar which is twice the sum of the principal curvatures is positive and vanishes on the dashed line CE. In the domain ACE the Ricci scalar is negative.

Beatrice explains the mystery to Dante's satisfaction, if not to the reader's, and goes on to catalogue the ORDERS OF THE ANGELS.

(translated by John Ciardi) [1]

The stretched S^3–s come in all sizes since their volume can take any positive value. Their intrinsic shape depends on two parameters presenting us with the intriguing problem to visualize the simplest symmetric forms of a compact Riemannian 3-space. This problem is all the more challenging because one is dealing here with mysterious objects of great mathematical beauty. While a hyperellipsoid with equation

$$(x/a)^2 + (y/b)^2 + (z/c)^2 + (u/d)^2 = 1 \tag{6.1}$$

embedded into the R^4 with different length of the axes $2a, 2b, 2c, 2d$ has only a few discrete symmetries – the stretched S^3–s are all homogeneous. That means they allow ∞^3 symmetry operations. In the symmetric case they have rotational symmetry in each point giving even ∞^4.

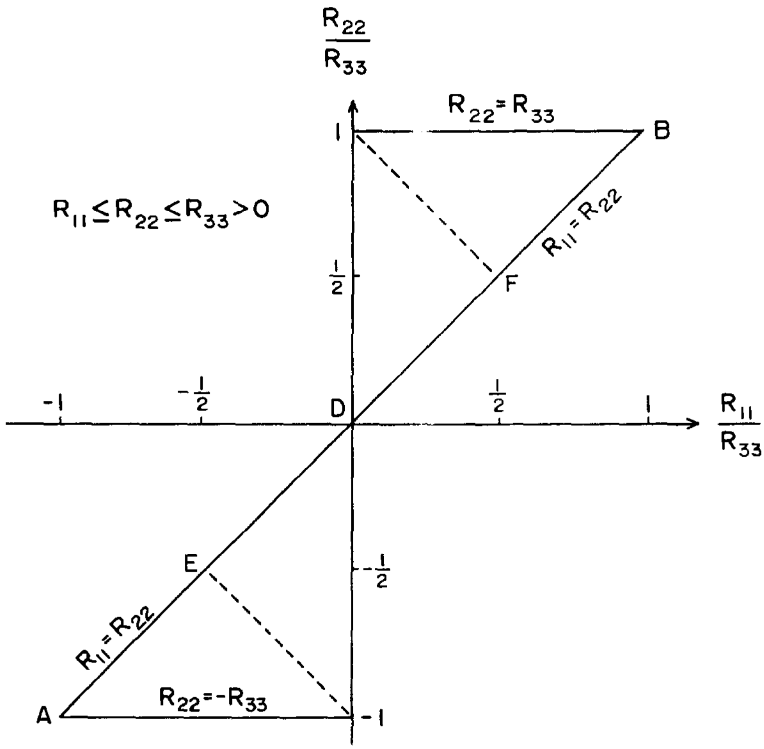

Fig. 3. Diagram for the eigenvalue ratios of the Ricci tensor in a stretched S^3. Point B denotes again the isotropically stretched S^3. Here $R_{11} = R_{22} = R_{33}$, the eigenvalues of the Ricci tensor are equal and thus the principal curvatures. The segment from -1 to 1 of the R_{22}/R_{33}-axis, except the origin D, does not represent possible configurations. While the map from the triangle ABC in Fig. 2 to the two triangles here is one-to-one with corresponding points denoted by the same letters, the line CD in Fig. 2 collapses here into the point D. If $R_{11} = 0$ then R_{22} has to vanish too.

Moreover, all finite subgroups of the $Sp(1) \cong SU(2)$ group of left translations can be used to identify points giving rise to different space forms for the stretched S^3. The examples are the lens spaces $L(p, p-1)$. It would seem apt to have a simple name for these intriguing models of space other than the bland *Bianchi type* IX. We also see no merit in the practice of the mathematicians to coin words with prefixes *pseudo, semi, hemi, para, etc* . Since Dante appears to have been the first to visualize a compact 3-space with a rich structure we shall call these spaces after him *dantes*.

A study of dantes began in 1949 with the largely unpublished investigations of Kurt Gödel in Princeton [8]. He studied all nine Bianchi types as space models for Einstein's theory of gravitation but announced only a few results of his extensive calculations – mostly without proof. His main concern was to demonstrate the existence of closed time-like world lines in rotating models of the universe.

In 1952 Abraham Taub discussed vacuum solutions of the Einstein field equations based on all nine Bianchi types [33]. Since then the study of dantes as spaces for cosmological models has been a major field of research in general relativity and cosmology. Principal insights have been obtained into the nature of space–time through the work of Charles Misner and it was shown that Mach's principle is not a consequence of Einstein's theory of gravitation [22].

But in spite of an extensive literature that may comprize now about 10^3 papers and books [26] the nature of the dantes has remained still obscure. If we study curved manifolds intrinsically, we get a feel for it comparable to touching a surface but leaving us blind for seeing its global properties. The sphere, the ellipsoid or the S^3 are best appreciated when viewed from outside, that is to say, imbedded into a Euclidean space of higher dimension. It is for this reason that we report here a few results for the imbedding of dantes. The general problem has been solved for RP^3 [23].

7. Isometric imbedding of Riemannian manifolds into Euclidean spaces

In 1873 Ludwig Schläfli described the problem of a local isometric embedding for a Riemannian manifold M into an m-dimensional Euclidean space R^m [27]. The points of the R^m can be denoted by a position vector $\boldsymbol{y}$

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}. \tag{7.1}$$

The line element in R^m is given by the metric

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{y} \equiv \mathrm{d}y_l\,\mathrm{d}y_l, \quad l = 1, \ldots, m. \tag{7.2}$$

In the neighborhood of one of its points P the manifold M, assumed to be n-dimensional, can be referred to a coordinate chart with coordinates x^μ. Little Greek indices like μ run here from 1 to n and the point P may be the origin of the chart.

A Riemannian manifold M has a line element

$$\mathrm{d}s^2 = g_{\mu\nu}(x^\lambda)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \quad \mu, \nu, \lambda = 1, \ldots, n, \quad g_{\mu\nu} = g_{\nu\mu}, \tag{7.3}$$

with positive definite quadratic form whose coefficients $g_{\mu\nu}$ are assumed to be analytic functions in the neighborhood N of the origin.

A local analytic embedding for the neighborhood N is then given by m analytic functions

$$y_l = y_l(x^\lambda), \quad l = 1, \ldots, m, \quad \lambda = 1, \ldots, n, \tag{7.4}$$

of n independent variables. To assume that this map from N into the R^m is locally injective (different points go into different points) and smooth one requests that the Jacobian $m \times n$ matrix

$$(\partial y_1/\partial x^\mu) \tag{7.5}$$

has rank n. Through this procedure one has lifted a piece of the manifold M into the Euclidean R^m where one can now look at it in a familiar surrounding. An example of this approach was shown in (1.3) where we embedded a large piece of the S^3 into the Euclidean R^4.

The embedding is called isometric if Euclid's metric on R^m is pulled back through the embedding onto the manifold M. This means simply that the $\mathrm{d}s^2$ in (7.2) and (7.3) should be the same for all x^μ in N. We get thus Schläfli's equations

$$\mathrm{d}y_l\,\mathrm{d}y_l = \frac{\partial y_l}{\partial x^\mu}\frac{\partial y_l}{\partial x^\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{7.6}$$

or the system of $\frac{1}{2}n(n+1)$ partial differential equations

$$\frac{\partial y_l}{\partial x^\mu}\frac{\partial y_l}{\partial x^\nu} = g_{\mu\nu}(x^\lambda). \tag{7.7}$$

The $y_l(x^\lambda)$ are here the unknown functions that have to be found as solutions of the $\frac{1}{2}n(n+1)$ equations with given right-hand sides. Schläfli guessed right that one needs in general $\frac{1}{2}n(n+1)$ functions $y_l(x^\lambda)$ to solve the equations and thus $m \geq \frac{1}{2}n(n+1)$.

Elie Cartan proved this conjecture in 1927. A proof published in the previous year by Maurice Janet had not been quite convincing and was made rigorous by the Russian mathematician C. Burstin in 1931 [16].

While two-dimensional Riemannian (Gaussian) manifolds are locally just surfaces in R^3 as Ossian Bonnet recognized, for the higher-dimensional ones one needs a lot more space for local isometric embedding. The three-dimensional manifolds will require in general a six-dimensional Euclidean space to accomodate them locally without distortion.

The problem assumes new dimensions when one is more ambitious seeking *global* isometric embedding of the whole Riemannian manifold. Manifolds are described as a patchwork of local neighborhoods. Having a local embedding for each patch in the crazy quilt of a manifold creates a problem of stitching the image pieces smoothly together in the Euclidean embedding space. One may thus be forced into higher dimensions or get back to where one had been before intersecting the image from another patch. If such self-intersections for a global embedding are not ruled out one speaks of an immersion. A figure eight is an immersion of S^1 in R^2.

An example of a two-dimensional manifold that does not fit isometrically into the R^3 is the flat torus, the Clifford surface. As piece of the periodic plane (see Fig. 1) the flat torus has locally an isometric embedding into the R^2. But this torus experiences terrible distortions and becomes afflicted by severe curvatures when it attempts to squeeze itself into Euclid's R^3. A plausible theorem states that there is no two-times continuously differentiable isometric embedding of the flat torus into the R^3. But there is the donut if one drops the adjective *isometric*.

Another celebrated example of an isometric global embedding is that of S^2 with opposite points identified and its natural metric – known also as RP^2, the projective plane. The Southern hemisphere without the equator, almost the whole space, is isometrically embedded into the R^3 just by *being there*, the identity map for the purists. But the opposite points at the equator that are here far apart and need to merge are now the problem. The global isometric embedding becomes possible in the R^5 by the map

$$\begin{aligned} &y_1 = x^2/\sqrt{2}, \quad y_2 = y^2/\sqrt{2}, \quad y_3 = z^2/\sqrt{2}, \\ &y_4 = xy, \quad y_5 = xz, \quad y_6 = yz. \end{aligned} \tag{7.8}$$

It immerses the unit sphere S^2

$$x^2 + y^2 + z^2 = 1 \tag{7.9}$$

into the hyperplane

$$y_1 + y_2 + y_3 = 1/\sqrt{2} \tag{7.10}$$

of the R^6 and thus into the R^5. The map is isometric since with $l = 1, \ldots, 6$

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}y_l\,\mathrm{d}y_l \\ &= [x^2 + y^2 + z^2][\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] + [x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z]^2 \end{aligned} \tag{7.11}$$

becomes

$$\mathrm{d}s^2 = \mathrm{d}y_l\,\mathrm{d}y_l = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{7.12}$$

on the S^2. Opposite points on S^2 with coordinates (x, y, z) and $(-x, -y, -z)$ are mapped into the same point of the R^5. It is easily seen that the map of the RP^2 into the R^5 is injective. This embedding of the projective plane is the famous Veronese surface. There is no two-times continuously differentiable isometric embedding of the RP^2 with the sphere metric into a Euclidean R^4.

From theorems like these it becomes clear that the global isometric embedding, even of simple manifolds, might be a very difficult problem. One could doubt whether it always had a solution. In 1956 John Forbes Nash laid these doubts to rest. In his epochal papers [19] he showed that global isometric embeddings existed albeit in Euclidean manifolds of high dimensions. For the analytic embedding of three-dimensional compact analytic Riemannian manifolds he needed at least 30 dimensions. Later work by Mikhael Gromov and Vladimir Rokhlin [9] reduced the numbers of dimensions considerably. But their minimal number still remains uncomfortably high and the existence theorems are not much help in finding a global isometric embedding for a given Riemannian manifold.

An isometric embedding creates a concrete model of an abstract manifold. But this model can only be useful for the study of the manifold if it turns out to be a submanifold of the Euclidean embedding space. That means the topology of the manifold (the notion of nearness for its points) must agree with the topology induced on its image in the embedding space by the Euclidean notion of nearness in this space. If this condition is not met, we see a mathematical horror picture show. Example: The immersion of the line $\phi_2 = \sqrt{2}\,\phi_1$ on the torus described in Fig. 1. The infinite one-dimensional straight line has been chopped into pieces that are completely irrationally arranged on the torus. The line comes infinitely close to every point on the bagel without covering it. To study a line in such a maimed form would be in the medical vocabulary a "suboptimal procedure". A further requirement is the rigidity of the embedding. That is, it must be fixed up to the congruence groups of the embedding space, i.e. its rigid motions. Naturally, one would like to keep the number of dimensions for the embedding Euclidean space as low as possible. We turn now to some embeddings of dantes.

8. The embedding of dantes

To find an embedding for the dantes one has to solve Schläfli's equations (7.7). This turns out to be a difficult problem and we shall not even attempt to cite those formulae here. Naturally, one will first try to embed the dantes as hypersurfaces into four dimensions since one may vaguely think of them as some hyper-ellipsoids. The answer is surprising: the only dante that can be isometrically embedded into the Euclidean R^4 is the S^3, the undistorted sphere.

After this disappointing result one tries to embed the dantes isometrically into the five-dimensional Euclidean space. In fact it turns out to be simpler to ask specifically whether one can embed a dante into an S^4. Although pieces of dantes could be found undisturbed in the S^4 thus leading to local isometric embedding, the result was disappointing again: no dante (except the S^3) can be embedded analytically and isometrically into an S^4 [20].

At this stage of the investigation it became clear that one should tackle the much more difficult problem of finding dantes that could be embedded into an S^5. The result of long and elaborate calculations was: There are no analytic isometric embeddings of a dante (except S^3) into the five-dimensional sphere [21]. The global embedding problem for the dantes is still unsolved. At the root of it lies a singular stability and rigidity of the S^3 against the homogeneous distortion that leads to the dantes.

However, in the search for an embedding, local embeddings were discovered. Although we have not yet been able to view the dantes *from the outside*, we can now look at well-defined pieces of them, some space forms of the dantes.

9. Forms of the dantes

Killing's search for the possible forms of spaces with constant curvature can also be extended to the dantes. All discrete subgroups of the left translations given by SU_2 give rise to a space form for the dante through point identification. The discrete subgroups of SU_2, discovered by Felix Klein [14], are double versions of the well-known cyclic, di-, tetra-, octa-, and icosahedral groups which occur as finite subgroups of SO_3. Hans Bethe discussed these double groups and their representations in 1929 [3] to account for the behavior of spinning electrons in crystal fields.

Since the group SU_2 is also the group of unit quaternions we can picture the elements of a discrete subgroup of order n as n points on a unit S^3. We describe the points of a dante through quaternions ξ with

$$\xi\,\bar{\xi} = R^2, \tag{9.1}$$

and take the unit quaternions λ_j $(j = 1, \dots, n)$

$$\lambda_j \bar{\lambda}_j = 1, \quad \text{no summation over } j,$$

as the elements of the discrete subgroup H_n of SU_2. The orbit of the subgroup H_n through the point ξ is then given by the set of n points ξ_j

$$\xi_j = \lambda_j \xi, \quad \xi_1 = \lambda_1 \xi, \quad \lambda_1 = 1. \tag{9.2}$$

The n points ξ_j for any ξ in (9.1) are then identified as one point of the quotient manifold S^3/H_n. The group H_n becomes then Poincaré's fundamental group for the dante form. Since this group is invariant it helps to characterize the dante form topologically.

The simplest example of a double group is one that doubles the unit element 1 of SO_3 to 1 and -1 of SU_2. With

$$H_2 = \{1, -1\} \tag{9.3}$$

we identify the opposite points

$$\xi_2 = (-1) \cdot \xi, \quad \xi_1 = 1 \cdot \xi \tag{9.4}$$

creating a "projective" dante half as large as its original.

The embedding of projective spaces is a fascinating chapter of geometry. We mention here only Francois Apéry's immersion of the projective plane (Boy's surface) [2] into the R^3 and the isometric embedding described in Section 7.

The projective dante with

$$a = b = 1, \quad c = 2 \tag{9.5}$$

can be isometrically embedded into the five-dimensional sphere S^5 by the "spin-1" map. The projective dante is given by the equation

$$|z_1|^2 + |z_2|^2 = 1, \quad z_1, z_2 \in C, \tag{9.6}$$

with points (z_1, z_2) and $(-z_1, -z_2)$ identified. Borrowing terminology from quantum mechanics we define the spin-j map $C^2 \to C^{2j+1}$ by

$$\zeta_m \equiv |j, m\rangle = \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{1/2} z_1^{j+m} z_2^{j-m}, \quad j \geq m \geq -j. \tag{9.7}$$

Here $2j$ is a positive integer while m runs through the values

$$j,\ j-1,\ \dots,\ -j.$$

The spin-$\frac{1}{2}$ map is the identity

$$\zeta_{1/2} \equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle = z_1, \qquad \zeta_{-1/2} \equiv |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = z_2. \tag{9.8}$$

The spin-1 map reads

$$\zeta_1 \equiv |1, 1\rangle = z_1^2, \quad \zeta_0 \equiv |1, 0\rangle = \sqrt{2} z_1 z_2, \quad \zeta_{-1} \equiv |1, -1\rangle = z_2^2. \tag{9.9}$$

The spin-j map gives with the binomial theorem

$$\sum_{m=-j}^{j} \bar{\zeta}_m \zeta_m = \sum_{-j}^{+j} \binom{2j}{j+m} (|z_1|^2)^{j+m} (|z_2|^2)^{j-m} = (|z_1|^2 + |z_2|^2)^{2j} = 1, \quad (9.10)$$

and provides thus an image of the S^3/Z_{2j} in the unit S^{4j+1}. This is clear if one considers that the ζ_m are the coordinates of a complex space of dimension $2j+1$ and thus of a real space with dimension $4j+2$. Eq. (9.10) gives the equation of a sphere in this space. Since the map employs homogeneous polynomials of degree $2j$ it is evident that the points (z_1, z_2) and

$$\left(z_1 \exp \frac{2\pi \mathrm{i} \mu}{2j},\ z_2 \exp \frac{2\pi \mathrm{i} \mu}{2j} \right), \quad 0 \leq \mu < 2j, \quad (9.11)$$

have the same image. We are thus embedding the lens dante $L(2j, 1)$ into the sphere S^{4j+1}.

From the spin-1 map we find from (9.9)

$$\mathrm{d}s^2 = \sum_{m=-j}^{j} |\mathrm{d}\zeta_m|^2 = 2|z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1|^2 + 4|\bar{z}_1\,\mathrm{d}z_1 + \bar{z}_2\,\mathrm{d}z_2|^2 \quad (9.12)$$

which can be written as

$$\mathrm{d}s^2 = 2(\omega_1^2 + \omega_2^2) + 4\omega_3^2. \quad (9.13)$$

We have put here

$$\mathrm{i}\omega_+ = \mathrm{i}(\omega_1 + \mathrm{i}\omega_2) = z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1, \quad \mathrm{i}\omega_3 = \bar{z}_1\,\mathrm{d}z_1 + \bar{z}_2\,\mathrm{d}z_2. \quad (9.14)$$

To see that ω_+ and ω_3 are left-invariant differential forms on SU_2 we write the elements U of SU_2 as

$$U = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \qquad \det U = |z_1|^2 + |z_2|^2 = 1. \quad (9.15)$$

The left-invariant differential forms become

$$\begin{aligned} U^+\,\mathrm{d}U &= \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} \mathrm{d}z_1 & -\mathrm{d}\bar{z}_2 \\ \mathrm{d}z_2 & \mathrm{d}\bar{z}_1 \end{pmatrix} \\ &= \begin{pmatrix} \bar{z}_1\,\mathrm{d}z_1 + \bar{z}_2\,\mathrm{d}z_2 & \bar{z}_2\,\mathrm{d}\bar{z}_1 - \bar{z}_1\,\mathrm{d}\bar{z}_2 \\ z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1 & z_1\,\mathrm{d}\bar{z}_1 + z_2\,\mathrm{d}\bar{z}_2 \end{pmatrix} \end{aligned} \quad (9.16)$$

with

$$\bar{z}_1\,\mathrm{d}z_1 + \bar{z}_2\,\mathrm{d}z_2 + z_1\,\mathrm{d}\bar{z}_1 + z_2\,\mathrm{d}\bar{z}_2 = 0 \quad (9.17)$$

because of the second equation of (9.15). We write

$$U^+\,\mathrm{d}U = -\mathrm{i}\,\sigma_\mu\,\omega_\mu, \quad \mu = 1, 2, 3, \qquad \sigma_\mu \sigma_\nu = 1\,\delta_{\mu\nu} + \mathrm{i}\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda, \quad (9.18)$$

where ω_μ are the left-invariant differential forms [1] and the σ_μ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.19}$$

The Eq. (9.18) means

$$U^+ \,\mathrm{d}U = -\mathrm{i} \begin{pmatrix} \omega_3 & \omega_1 - \mathrm{i}\,\omega_2 \\ \omega_1 + \mathrm{i}\omega_2 & -\omega_3 \end{pmatrix}. \tag{9.20}$$

This gives for the natural metric on SU_2

$$\mathrm{d}l^2 = \det U^+ \,\mathrm{d}U = (|z_1|^2 + |z_2|^2)(|\,\mathrm{d}z_1|^2 + |\,\mathrm{d}z_2|^2) = \omega_1^2 + \omega_2^2 + \omega_3^2. \tag{9.21}$$

Comparing with (9.13) we see that the embedded projective dante is a P^3 with its natural metric stretched by a factor 2 in the 3-direction. The projective dante is given by the equations

$$|\zeta_1|^2 + |\zeta_0|^2 + |\zeta_{-1}|^2 = 1, \qquad 2\zeta_1\,\zeta_{-1} - \zeta_0^2 = 0. \tag{9.22}$$

Introducing new variables

$$\zeta_1 = \frac{1}{\sqrt{2}}(\zeta_+ + \zeta_-) \quad \zeta_{-1} = \frac{1}{\sqrt{2}}(\zeta_+ - \zeta_-), \tag{9.23}$$

we can write Eq.(9.22) as

$$|\zeta_+|^2 + |\zeta_0|^2 + |\zeta_-|^2 = 1 \quad \zeta_+^2 + \zeta_-^2 - \zeta_0^2 = 0. \tag{9.24}$$

The projective dante is the complex projective circle on the real unit S^5. This most harmonious object represents the point D in Figs. 2 and 3.

While the Veronese embedding of the RP^3, the analog of (7.8), needs the vastness of an S^8 a simple stretch along one axis by a factor of $\sqrt{2}$ enables us now to squeeze the projective dante elegantly into an S^5.

The embedding of the projective dante gives as a spin-off an embedding of the projective plane. By taking z_2 real we obtain a plane and parametrize the spin-1 map

$$\begin{aligned} &z_1 = \sin\vartheta \mathrm{e}^{\mathrm{i}\varphi} \quad z_2 = \cos\vartheta \\ &\zeta_1 = \sin^2\vartheta \mathrm{e}^{2\mathrm{i}\varphi}, \quad \zeta_0 = \sqrt{2}\sin\vartheta\cos\vartheta \mathrm{e}^{\mathrm{i}\varphi}, \quad \zeta_{-1} = \cos^2\vartheta. \end{aligned} \tag{9.25}$$

[1] To check the normalization of the left-invariant differential forms ω_μ we write the Maurer–Cartan equations in agreement with (5.4)

$$\begin{aligned} \mathrm{d}(U^+\,\mathrm{d}U) &= -\mathrm{i}\sigma_\mu\,\mathrm{d}\omega_\mu = \mathrm{d}\,U^+ \wedge \mathrm{d}U = \mathrm{d}\,U^+ \wedge UU^+\mathrm{d}U = \mathrm{d}\,U^+U \wedge U^+\,\mathrm{d}U \\ &= -\,U^+\,\mathrm{d}U \wedge U^+\,\mathrm{d}U = \sigma_\lambda\omega_\lambda \wedge \sigma_\nu\omega_\nu = \mathrm{i}\,\varepsilon_{\mu\lambda\nu}\sigma_\mu\omega_\lambda \wedge \omega_\nu, \end{aligned}$$

or

$$\mathrm{d}\omega_\mu = -\varepsilon_{\mu\lambda\nu}\omega_\lambda \wedge \omega_\nu.$$

We obtain then for RP^2 the metric

$$ds^2 = \sum_{m=-j}^{j} |\,d\zeta_m|^2 = 2\,[d\vartheta^2 + \sin^2\vartheta\,(1 + \sin^2\vartheta)\,d\varphi^2]. \tag{9.26}$$

The embedding of the projective plane goes into the S^4 like the Veronese surface.

10. Spherical space forms embedded

When one is trying to look at the mysterious spherical space forms from the outside one wants to find an isometric embedding for them. A first guess was that it should be simplest to embed the space forms of the isotropic S^3. But it turned out to be a very difficult problem that has yielded its solution only in the last years.

The critical idea was to look for embedding into Euclidean spheres and to require that the embedded manifold has a three-dimensional volume that is stationary (a minimum) under a variation of the isometric embedding. A breakthrough for the solution for this much more restricted problem, the isometric embedding as a minimal manifold of a higher-dimensional sphere, was achieved by Tsunero Takahashi in 1966 [32]. He found that such embeddings are obtained by means of functions which are all eigenfunctions of the Laplace operator on the manifold belonging to the same eigenvalue of the operator. Using Takahashi's theorem Dennis DeTurck and Wolfgang Ziller managed in 1992 to find minimal isometric embeddings into spheres for all homogeneous spherical space forms [6].

Unfortunately, in spite of all efforts to keep the dimensions of their engulfing space as low as possible, there were none or them that fitted into R^6 – apart from the trivial map into R^4. The pièce de résistance of their impressive collection had been first shown to exist by Norio Ejiri in 1981 [7] and was actually constructed by Katsuya Mashimo in 1984 [18]. The embedding is given by

$$\begin{aligned}
y_1 + iy_2 &= \tfrac{1}{4}\sqrt{6}\,(\bar z_1 \bar z_2^5 - \bar z_1^5 \bar z_2),\\
y_3 + iy_4 &= \tfrac{1}{4}\bar z_2^4(5|z_1|^2 - |z_2|^2) + \tfrac{1}{4}\bar z_1^4(5|z_2|^2 - |z_1|^2),\\
y_5 + iy_6 &= \tfrac{1}{4}\sqrt{10}\,[z_1 \bar z_2^3(2|z_1|^2 - |z_2|^2) + \bar z_1^3 z_2(|z_1|^2 - 2|z_2|^2)],\\
y_7 &= (\sqrt{15}/4i)(|z_1|^2 - |z_2|^2)(z_1^2 \bar z_2^2 - \bar z_1^2 z_2^2)
\end{aligned} \tag{10.1}$$

with

$$|z_1|^2 + |z_2|^2 = 1 \rightarrow y_\alpha y_\alpha = \tfrac{1}{16}, \quad \alpha = 1, \ldots, 7. \tag{10.2}$$

DeTurck and Ziller found that this isometric immersion of S^3 into S^6 is actually the embedding of the space form S^3/T^*, where T^* is the tetrahedral double group represented by the 24 unit quaternions

$$\pm 1, \quad \pm i, \quad \pm j, \quad \pm k, \quad \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k). \tag{10.3}$$

We had already noticed in Section 9 that the projective non-spherical dante was nestling snugly in the S^5 while the Veronese sphere needed an S^8 as lebensraum. We shall study in the following a similar case.

11. Poincaré's cube

Another double group of interest is Hamilton's H_8. Its eight elements can be represented by the quaternions

$$H_8 = \{\pm 1,\ \pm\mathrm{i},\ \pm\mathrm{j},\ \pm\mathrm{k}\}. \tag{11.1}$$

This group is obtained through the doubling of the Klein 4-group acting as rotations about three orthogonal axes by π, plus, the identity. It is the smallest non-Abelian group whose subgroups are all normal.

When Poincaré studied the fundamental group of manifolds – now known as the one-dimensional homotopy group – he investigated the possible identifications of vertices, edges and faces of a cube. There was the well-known example of the periodic cube whose identifications are brought about by finite translations. Henri Poincaré demonstrated in his paper, [24] which introduced modern topology, a cube with a different topology. He identified opposite faces of an oriented cube by combining a translation with a rotation by $\frac{1}{2}\pi$.

Poincaré's scheme can be elegantly described in a unit S^3 whose points are unit quaternions ξ

$$\xi\,\bar{\xi} = 1. \tag{11.2}$$

The 16 quaternions (belonging to the Hurwitz ring of integral quaternions)

$$\xi = \tfrac{1}{2}(\pm 1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}) \tag{11.3}$$

lie on the S^3 and form the vertices of a hypercube inscribed in the S^3. In one dimension lower, this corresponds to inscribing a cube in the S^2 as Kepler did into the sphere of Saturn. The sunlight projects the 12 edges of the cube as great circle arcs on Saturn's orb. They divide the sphere into six quadrangles meeting in threes at each of the eight corners (see Fig. 4).

Now back to the hypercube. One of its eight hypersurfaces is the cube C_1 with eight vertices $V(C_1)$

$$V(C_1) = \tfrac{1}{2}(1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}). \tag{11.4}$$

Through left translations by multiplication with the eight elements of H_8 one obtains eight cubes

$$1C_1, -1C_1, \mathrm{i}C_1, -\mathrm{i}C_1, \mathrm{j}C_1, -\mathrm{j}C_1, \mathrm{k}C_1, -\mathrm{k}C_1. \tag{11.5}$$

The six faces of the cube C_1 are given by the vertices

$$\begin{array}{lll} \tfrac{1}{2}(1, \mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}), & \tfrac{1}{2}(1, -\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}), & \tfrac{1}{2}(1, \pm\mathrm{i}, \mathrm{j}, \pm\mathrm{k}), \\ \tfrac{1}{2}(1, \pm\mathrm{i}, -\mathrm{j}, \pm\mathrm{k}) & \tfrac{1}{2}(1, \pm\mathrm{i}, \pm\mathrm{j}, \mathrm{k}), & \tfrac{1}{2}(1, \pm\mathrm{i}, \pm\mathrm{j}, -\mathrm{k}). \end{array} \tag{11.6}$$

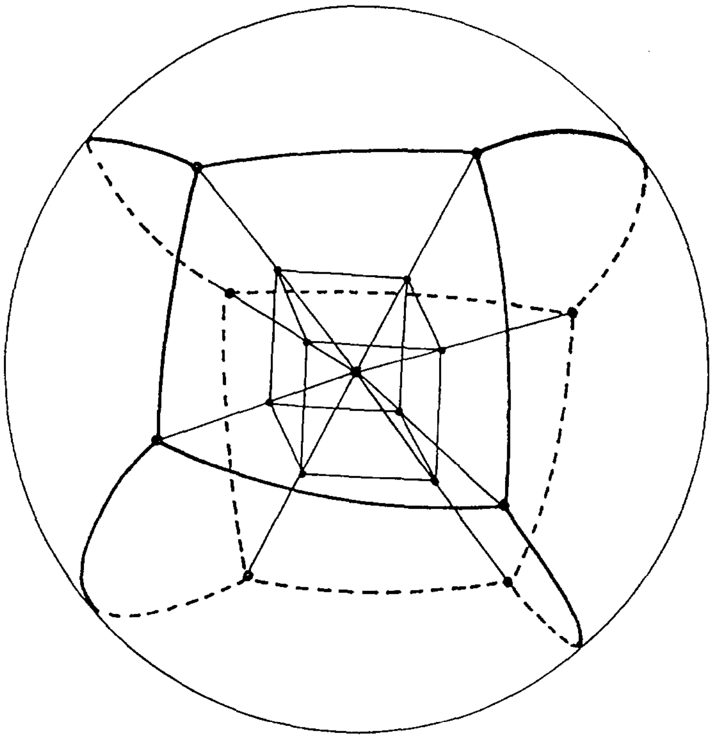

Fig. 4. The central projection of a cube onto a concentric sphere creates a division of the S^2 into six congruent quadrangles bounded by arcs of great circles [after M. Berger, *Geometry II*. (Springer, New York, 1987) p. 45, with permission].

The four points on each face lie on a hyperplane through the origin. If we name the four coordinates of ξ,

$$\xi = u \cdot 1 + x \cdot \mathrm{i} + y \cdot \mathrm{j} + z \cdot \mathrm{k}, \tag{11.7}$$

the faces of the cube C_1 lie on the hyperplanes

$$u = \pm x, \quad u = \pm y, \quad u = \pm z \tag{11.8}$$

through the origin $\xi = 0$. The six hyperplanes cut S^3

$$u^2 + x^2 + y^2 + z^2 = 1 \tag{11.9}$$

in great spheres S^2 of radius 1 in analogy with the lower-dimensional example where the planes going through the edges and the center of the cube cut out arcs of great circles on S^2. The cube C_1 thus gives rise to a cubicle on S^3 whose six walls are quadrangles of great spheres with radius 1 (see Fig. 5). By left translation of this cubicle with the elements of H_8 we disect thus the S^3 into eight cubicles.

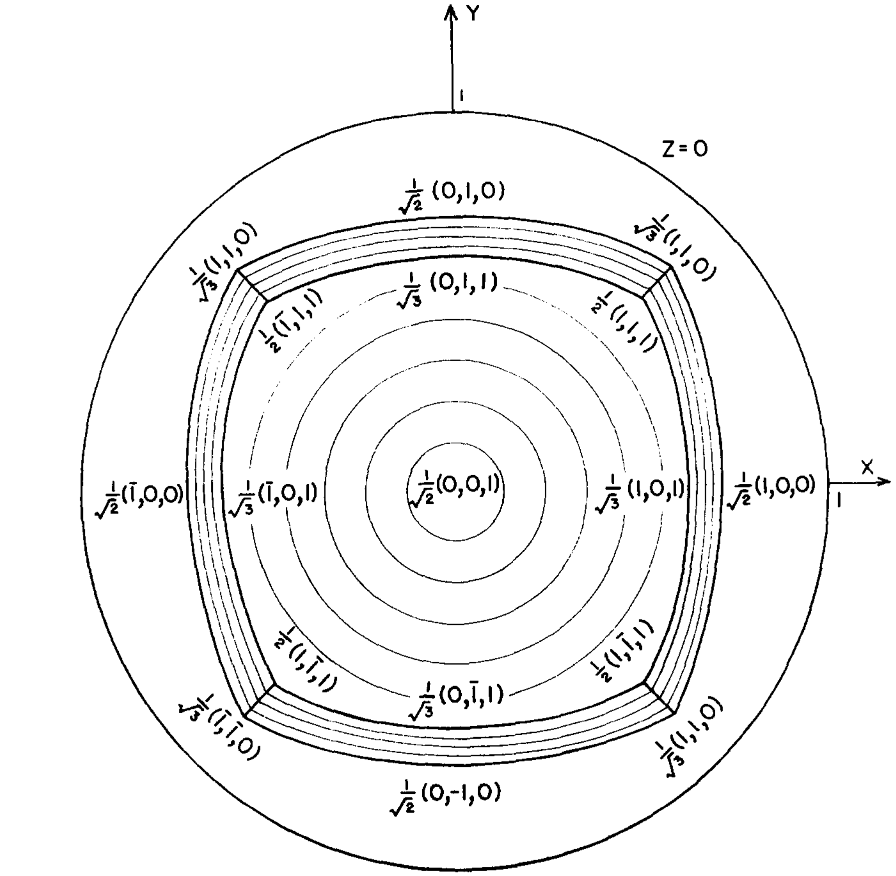

Fig. 5. One of the eight cubicles in the S^3 parallely projected on the hyperplane $u = 0$. The view is from the positive z-axis. We look down on the bulging top face of the cubicle centered on the point $x = y = z = 0$, $u = 1$ of the S^3 and see half of its four side faces. The faces of the projected cubicle are given by the intersection of the three ellipsoids of rotation $2x^2 + y^2 + z^2 = x^2 + 2y^2 + z^2 = x^2 + y^2 + 2z^2 = 1$. The coordinates of the visible vertices, mid-edges and face centers are given following crystallographer's convention by indicating a negative value of a coordinate by a bar over the number. The u-values of all points of the cubicle are given by $u = +(x^2 + y^2 + z^2)^{1/2}$.

The cubicle $\hat{C}_1$ with the eight vertices $\frac{1}{2}(1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k})$ has as its center the quaternion 1 on the S^3. This center is obtained as $\frac{1}{4}$ the sum of the $\hat{C}_1$ vertices. We have then that the centers of the eight cubicles are also the elements of the group H_8, namely $\{\pm 1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}\}$. These eight points form the vertices of a regular 16-cell dual to the hypercube.

Each cubicle has an "anticubicle" whose eight vertices are the eight other points with the opposite values of the coordinates. The anticubicle is thus simply obtained by multiplication with -1. In a projective space, like the orthogonal group $SO(3)$, each cubicle would coincide with its anticubicle.

We see from Fig. 6 that a cubicle has six neighbors sharing one of its six faces with one of each of the neighbors. That means: cubicles have either four vertices in common or none.

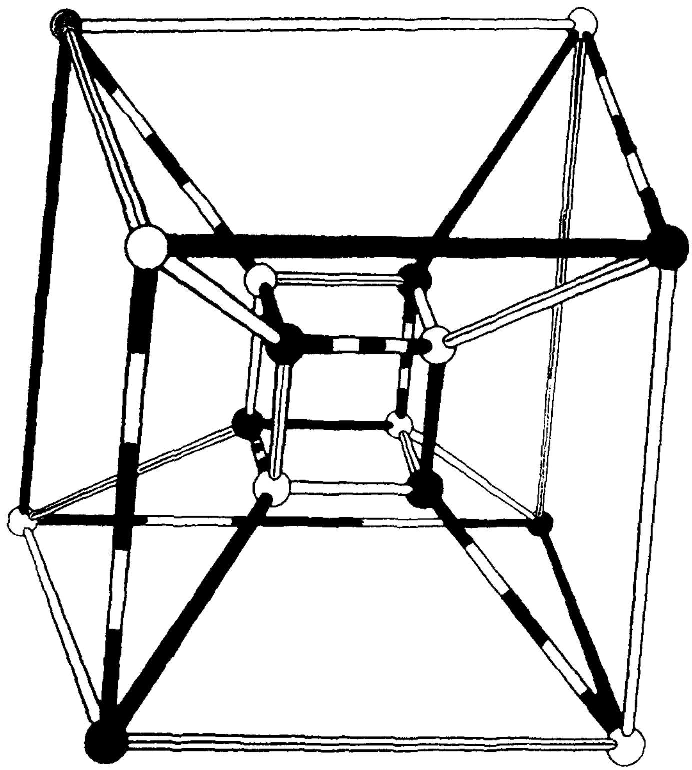

Fig. 6. Looking at a transparent cube from outside we may see its rear face framed inside its front face. Here we look at the hypercube from the outside and see its *rear* hyperface, the small cube inside the *front* hyperface, the large cube. The six truncated Egyptean pyramids connecting the rear hyperface with the front hyperface are the six other hyperfaces (cubes) of the hypercube. One is asked to imagine that the interior of the front hyperface is a volume separate from that of the seven other hyperfaces. This is analogous to realizing that the front face of a box as window covers a different area from that of the rear-end and side faces. Vertices and edges to be identified in the cubicle under the group H_8 (see text) have been marked identically [after D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (Chelsea, New York, 1952) p. 150, with permission].

In the last case they are antipodes of each other. Every edge is shared by three cubicles and at each corner four cubicles abut. We obtain thus for the Euler characteristic of S^3 as the alternating sum of

$$\text{vertices} - \text{edges} + \text{faces} - \text{cubicles} = 16 - 8 \times \tfrac{12}{3} + 8 \times \tfrac{6}{2} - 8 = 0.$$

We consider now the bottom quadrangle of the cubicle $\hat{C}_1$ spanned by the vertices

$$\begin{aligned} \hat{C}_1^1 &= \tfrac{1}{2}(1+\mathrm{i}+\mathrm{j}-\mathrm{k}), & \hat{C}_1^2 &= \tfrac{1}{2}(1-\mathrm{i}+\mathrm{j}-\mathrm{k}), \\ \hat{C}_1^3 &= \tfrac{1}{2}(1-\mathrm{i}-\mathrm{j}-\mathrm{k}), & \hat{C}_1^4 &= \tfrac{1}{2}(1+\mathrm{i}-\mathrm{j}-\mathrm{k}). \end{aligned} \tag{11.10}$$

The scalar product of $\hat{C}_1^1$ with $\hat{C}_1^2$ is equal to $\frac{1}{2}$. This means that the arc of the edge $\hat{C}_1^1\hat{C}_1^2$ in the bottom quadrangle is 60° long since $\cos^{-1}\frac{1}{2} = 60°$. The scalar product of $\hat{C}_1^1$

with $\hat{C}_1^3$ vanishes. We learn from this that the diagonal of the quadrangle has a length of 90°.

The top quadrangle of the cubicle $\hat{C}_1$ is spanned by the vertices

$$\begin{aligned}\hat{C}_1^5 &= \tfrac{1}{2}(1+\mathrm{i}+\mathrm{j}+\mathrm{k}), &\quad \hat{C}_1^6 &= \tfrac{1}{2}(1-\mathrm{i}+\mathrm{j}+\mathrm{k}),\\ \hat{C}_1^7 &= \tfrac{1}{2}(1-\mathrm{i}-\mathrm{j}+\mathrm{k}), &\quad \hat{C}_1^8 &= \tfrac{1}{2}(1+\mathrm{i}-\mathrm{j}+\mathrm{k}).\end{aligned} \tag{11.11}$$

Left multiplication with the element $k \in H_8$ moves the bottom quadrangle onto the top quadrangle of the cubicle $\hat{C}_1$. We have in fact

$$\hat{C}_1^1 \to \hat{C}_1^6,\ \hat{C}_1^2 \to \hat{C}_1^7,\ \hat{C}_1^3 \to \hat{C}_1^8,\ \hat{C}_1^4 \to \hat{C}_1^5. \tag{11.12}$$

We can also carry out this motion less abruptly by multiplying from the left with

$$\mathrm{e}^{k\varphi} = 1\cos\varphi + k\sin\varphi, \quad 0 \le \varphi \le \tfrac{1}{2}\pi, \tag{11.13}$$

and let φ grow from zero to $\frac{1}{2}\pi$. The vertices of the bottom quadrangle move then along the (Clifford left parallel) diagonals of the four side quadrangles into the vertices of the top quadrangle. The center of the bottom quadrangle lies at $(1-k)/\sqrt{2}$, that of the top quadrangle is situated at $(1+k)/\sqrt{2}$. The geodesic connecting the two centers runs through the point 1 which is the center of the cubicle. This great circle arc is traced by multiplying $(1-k)/\sqrt{2}$ with the (11.13) factor $\mathrm{e}^{k\varphi}$ from the left. It puts the positive k-axis into the cubicle. Its length from bottom to top is 90°. The four diagonals of (11.12) on the four sides are left-parallels of the k-axis. The left-multiplication with k thus brings about a screw motion propelling the bottom of the cubicle $\hat{C}_1$ onto the top while rotating it by 90° right-handedly.

The same procedure can be carried out with the cubicle $\hat{C}_1$ by left-multiplication with j and with i. We then end up with 12 diagonals of the six faces of $\hat{C}_1$ intersecting each other at right angles at the centers of the quadrangles. The centers of the opposite faces are pierced by the three axes through the centers of the cubicle. We have thus three systems of Clifford left parallels that intersect each other orthogonally in the center of the cubicle and in the midpoints of the six faces.

Now comes the crucial step of forming the quotient space S^3/H_8. The S^3 that had been filled with eight cubicles is thus reduced to the single cubicle $\hat{C}_1$ since all the others have been obtained from it by left-translation with the elements of the quaternion group H_8. This single cubicle has only three faces since the opposite ones have been identified after a right-handed rotation of 90° (see Fig. 7). Each quadrangle has four different edges which are the same for each face. Of the eight vertices only two remain different. The different points are connected by four edges like the north and the south pole of the earth by four meridians. The diagonals of the faces are now circles connecting a vertex with itself. This is Poincaré's magic cubicle, an object of exquisite symmetry and beauty. Its fundamental group is H_8 which shows that the 3-manifold is not homeomorphic to any of Killing's lens spaces. Heinz Hopf, the great Swiss topologist, identified it in his thesis [10] as a possible space form of the three-dimensional space of positive curvature – a homogeneous universe reduced to one eighth of its potential volume.

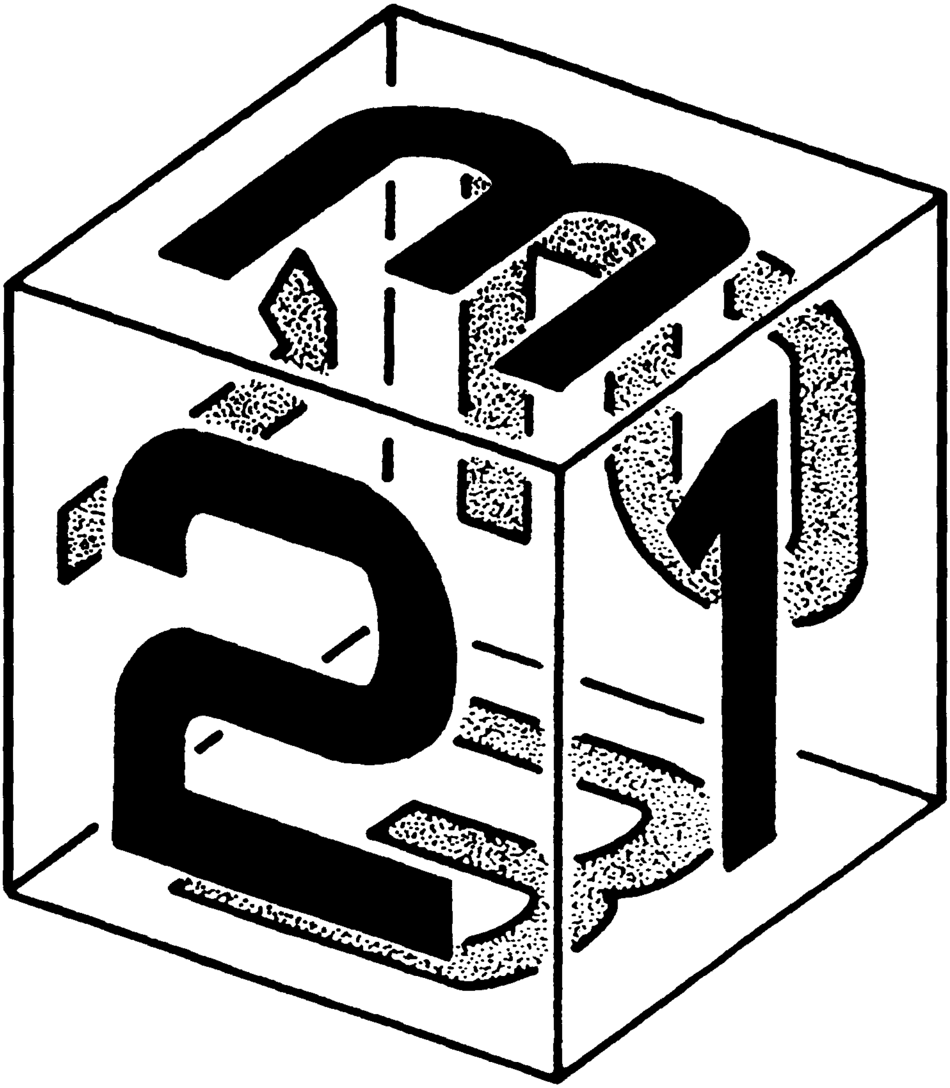

Fig. 7. Opposite faces of the cubicle are identified under the right-handed rotation by 90°. Poincaré's identification is shown for the faces of the cube [from Jeffrey R. Weeks, *The Shape of Space* (Marcel Dekker, New York, 1985) p. 228, with permission].

12. Embedding of the Poincaré cubicle

Besides the spherical space S^3, the projective space RP^3, and the periodic cube, the Poincaré cubicle is the simplest compact form of space. It is a most beautiful and harmonious construct that should appear in all geometry books. While it can be studied as a quotient space one would like to see its isometric embedding into a higher-dimensional Euclidean space.

The spaces S^3, RP^3, $(S^1)^3$ have the isometric embeddings

$$S^3: \qquad x_j x_j = 1 \subset R^4, \quad j = 1,2,3,4, \tag{12.1}$$

$$RP^3: \qquad y_j = \frac{1}{\sqrt{2}}(x_j)^2, \quad y_{kl} = x_k x_l, \quad 1 \le k < l \le 4, \quad x_j x_j = 1, \tag{12.2}$$

$$(S^1)^3: \qquad x_\alpha + \mathrm{i} y_\alpha = \mathrm{e}^{\mathrm{i}\phi_\alpha}, \quad \alpha = 1,2,3, \quad \phi_\alpha \in R. \tag{12.3}$$

An isometric embedding of the Poincaré cubicle was given by DeTurck and Ziller [6]. It is a minimal embedding into S^8 of radius $(3/80)^{1/2}$. However, a one-parameter set of dantes with the topology of the Poincaré cubicle has been isometrically embedded into R^5 [20]. The dantes have vanishing Ricci scalar and the metric

$$\mathrm{d}s^2 = \tfrac{1}{4}(1+\lambda)^2\omega_1^2 + \tfrac{1}{4}(1-\lambda)^2\omega_2^2 + \omega_3^2. \tag{12.4}$$

The embedding into the R^5 is given by

$$\begin{aligned} y_1 + \mathrm{i} y_2 &= \tfrac{1}{2}[z_1^4 + z_2^4] - \lambda z_1^2 z_2^2 \\ y_3 + \mathrm{i} y_4 &= z_1^3 \bar{z}_2 - \bar{z}_1 z_2^3 + \lambda z_1 z_2 (|z_1|^2 - |z_2|^2) \\ y_5 &= \tfrac{1}{2}\sqrt{3}\,[z_1^2 \bar{z}_2^2 + \bar{z}_1^2 z_2^2 - \tfrac{1}{3}\lambda[(|z_1|^2 - |z_2|^2)^2 - 2|z_1|^2|z_2|^2]] \end{aligned} \tag{12.5}$$

with

$$|z_1|^2 + |z_2|^2 = 1 \quad \text{and} \quad \lambda \text{ arbitrary } (|\lambda| \neq 1). \tag{12.6}$$

It follows that

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 = \tfrac{1}{4}(1 + \tfrac{1}{3}\lambda^2). \tag{12.7}$$

This shows that the embedded manifold fits into an S^4. It is easily checked that the substitutions

$$(z_1 \to \mathrm{i} z_1,\ z_2 \to -\mathrm{i} z_2) \quad \text{and} \quad (z_1 \to z_2,\ z_2 \to -z_1) \tag{12.8}$$

generate the quaternion group H_8 and leave the map (12.5) invariant. Since the map is isometric for the dante (12.4) it is regular and one finds without any major effort that it is indeed injective on Poincaré's cubicle. It is remarkable that this fascinating construct exists not only as quotient space but lives in various shapes of smoothly embedded dantes in the pelagic depth of the S^4 without intersecting itself in any way.

Suprisingly, these are precisely the dantes – apart from S^3 – that can be isometrically immersed into S^4.

We have a particularly pretty case for a vanishing parameter λ. The dante becomes then axially symmetric and slimmed, orthogonal to the axis, by a factor of 2. The metric is then

$$\mathrm{d}s^2 = \tfrac{1}{4}(\omega_1^2 + \omega_2^2) + \omega_3^2, \tag{12.9}$$

and the embedding is given by

$$y_1 + \mathrm{i}y_2 = \tfrac{1}{2}[z_1^4 + z_2^4], \quad y_3 + \mathrm{i}y_4 = z_1^3\bar{z}_2 - \bar{z}_1 z_2^3, \quad y_5 = \tfrac{1}{2}\sqrt{3}\,[z_1^2\bar{z}_2^2 + \bar{z}_1^2 z_2^2] \tag{12.10}$$

with

$$|z_1|^2 + |z_2|^2 = 1.$$

We have the nagging doubt that this is not yet the most elegant formulation. But quaternions suspected as vehicle for such an improvement have so far refused to admit that.

13. The anti-Mach universe

The idea that the earth's rotation against the stars might be indistinguishable from the opposite rotation of the stars against a fixed earth appears in the *The Science of Mechanics* by Ernst Mach [17]. When Einstein's special relativity was proposed in 1905 velocities of bodies larger than the speed of light were outlawed. The thought of stars moving with a million times the speed of light to making a day for a resting earth should then have appeared patently absurd to a relativist. Not so to Albert Einstein. In 1916 he created the theory of general relativity with the express wish of conceiving rotation also – like translation – as a relative motion of the body.

As a gift for Einstein's 70th birthday his friend Kurt Gödel demonstrated that general relativity conceives of a rotating universe that defies Mach's principle of the relativity of rotation. Einstein, however, did not appreciate Gödel's counterexample because the space of the Gödel cosmos was not compact and the space–time manifold allowed closed time-like world lines which Einstein deemed unphysical. However, a model of the cosmos with a compact space and without closed time-like world lines could be constructed no longer open to Einstein's objection. This anti-Mach universe was in a state of rotation against absolute space. It demonstrated that Einstein's wish to turn the rotation of a body into a relative motion with respect to the cosmos was *not* a consequence of his general relativity.

The anti-Mach universe is given by the line element [22]

$$\begin{aligned} \mathrm{d}s^2 = &- \mathrm{d}t^2 - R\sqrt{1-2k^2}\,\omega_3\,\mathrm{d}t \\ &+ (\tfrac{1}{2}R)^2[(1-k)\,\omega_1^2 + (1+k)\,\omega_2^2 + (1+2k^2)\,\omega_3^2], \end{aligned} \tag{13.1}$$

with parameters k and R such that

$$|k| \le \tfrac{1}{2}, \quad R > 0. \tag{13.2}$$

This line element is a solution of the Einstein field equations with incoherent matter, with a $\Lambda > 0$ term and

$$\frac{\kappa\rho}{2\Lambda} = 1 - 4k^2, \quad \Lambda = \frac{1}{R^2(1-k^2)},$$

where ρ is the density of the incoherent matter and κ is the gravitational constant. Investigating the properties of this model universe one can easily substantiate all the statements made above.

The t = const hypersufaces are given by the dante with the line element

$$\mathrm{d}s^2 = (\tfrac{1}{2}R)^2[(1-k)\,\omega_1^2 + (1+k)\,\omega_2^2 + (1+2k^2)\,\omega_3^2]. \tag{13.3}$$

This dante, endowed with the topology of RP^3, has the following embedding:

$$\begin{aligned}
y_1 + \mathrm{i}\,y_2 &= (R/2)\sqrt{((1-k)^2+k^2)/2}\,(z_1^2 - z_2^2),\\
y_3 &= (R/2)\sqrt{((1-k)^2+k^2)/2}\,(z_1\bar{z}_2 + \bar{z}_1 z_2),\\
y_4 + \mathrm{i}\,y_5 &= (R/2)\sqrt{((1+k)^2+k^2)/2}\,(z_1^2 + z_2^2).\\
y_6 &= (R/2)\sqrt{((1+k)^2+k^2)/2}\,(-\mathrm{i})(z_1\bar{z}_2 - \bar{z}_1 z_2),\\
y_7 + \mathrm{i}\,y_8 &= (R/2)\sqrt{(1-2k^2)/2}\,2z_1 z_2,\\
y_9 &= (R/2)\sqrt{(1-2k^2)/2}\,(2z_2\bar{z}_2 - 1).
\end{aligned} \tag{13.4}$$

Since

$$\sum_{k=1}^{9} y_k^2 = \frac{R^2}{8}(3+2k^2),$$

the embedding is actually into an eight-dimensional sphere ([23]).

14. Conclusion

Luigi Bianchi's discovery of the dantes in 1897 presented his fellow mathematicians with a cornucopia of beautiful spaces. Our attempt to grasp their geometry by looking at them from outside has only begun. In special cases has the first author been able to immerse them without distortion by finding all those that fit into the R^5 or S^5 [20,21]. These immersions turned out to embeddings of dante forms through simple functions that play an important rôle on these spaces. It is a challenging problem in "space research" to find through isometric embedding a proper setting for Bianchi's mathematical gems.

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